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Certain Properties of a Subclass of Non-Bazilevic Functions of Complex Order

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Abstract:
For function $f(z)$ of the form

$$f(z) = z + a_2 z^2 + \ldots$$

which are analytic and univalent in the open unit disk D. The authors define certain new subclass $S_{\lambda,l}^{m,\alpha}(A,B,b,\beta,\theta)$ of analytic functions which satisfy the condition that

$$1+\frac{1}{b}\left\{\frac{\beta e^{i\theta}\left(I^m(\lambda,l)f(z)^\alpha\right)' + z(1-\beta)\left[\left(I^m(\lambda,l)f(z)^\alpha\right)' + z\left(I^m(\lambda,l)f(z)^\alpha\right)''\right]}{\beta\left(I^m(\lambda,l)f(z)^\alpha\right) + (1-\beta)z\left(I^m(\lambda,l)f(z)^\alpha\right)'} - \alpha\right\} \prec \frac{1+Az}{1+Bz}$$

where b is any non-zero complex number and $\prec$ denote the subordination symbol. A and B are arbitrary constants with $-1 \le B < A \le 1$. Coefficient bounds, growth and distortion theorems for functions belonging to the said subclass $S_{\lambda,l}^{m,\alpha}(A,B,b,\beta,\theta)$ were determined.
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1. Introduction and Preliminary Definitions

Let S(p) denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad z \in U \qquad (1)$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$.

Supposing we pose an index α on (1) such that

$$f(z)^\alpha = \left(z + a_2 z^2 + \ldots\right)^\alpha$$

then

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1} \qquad z \in D \qquad (2)$$

see [4,5] for detailed expansion.

Using Aouf et al derivative operator [2], we can write for function $f(z)^\alpha$ defined in (2) that

$$I^m(\lambda,l)f(z)^\alpha = \left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha + \sum_{k=2}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m a_k(\alpha)z^k \quad (3)$$

$$m\in N_0,\ \alpha>0,\ \lambda\geq 0,\ l\geq 0 \quad and \quad z\in D.$$

Akbarally et al [1] among others had earlier considered the class $S_p(A,B,b,\lambda)$. However, for the sake of our present investigation, let $S_{\lambda,l}^{m,\alpha}(A,B,b,\beta,\theta)$ denote the subclass of S that consists of function $f(z)$ satisfying the condition that

$$1+\frac{1}{b}\left\{\frac{\beta e^{i\theta}\left(I^m(\lambda,l)f(z)^\alpha\right)' + z(1-\beta)\left[\left(I^m(\lambda,l)f(z)^\alpha\right)' + z\left(I^m(\lambda,l)f(z)^\alpha\right)''\right]}{\beta\left(I^m(\lambda,l)f(z)^\alpha\right)+(1-\beta)z\left(I^m(\lambda,l)f(z)^\alpha\right)'}-\alpha\right\} \prec \frac{1+Aw(z)}{1+Bw(z)}. \quad (4)$$

$$0\leq\beta\leq -1,\ \theta<\left|\frac{\pi}{2}\right|, m\in N_0,\ \alpha>0,\ \lambda\geq 0,\ l\geq 0 \quad and \quad z\in D$$

Next, the we considered the coefficient bounds for functions belonging to the class $S_{\lambda,l}^{m,\alpha}(A,B,b,\beta,\theta)$.

2. Main Result

Theorem 1: Let f be a function of the form (2), then $f(z)^\alpha \in S_{\lambda,l}^{m,\alpha}(A,B,b,\beta,\theta)$ if and only if

$$\frac{\sum_{k=2}^{\infty}\left[(\alpha+k-1)[\beta e^{i\theta}+(1-\beta)(\alpha+k-1)]+|M|\right]T_{K,\alpha}^m\left|a_k(\alpha)\right|}{|b|(A-B)\left[\beta+\alpha(1-\beta)-\alpha\beta(e^{i\theta}-1)(1+B)\right]}\leq 1$$

where

$$M = b(A-B)[\beta+(1-\beta)(\alpha+k-1)] - B(\alpha+k-1)[\beta e^{i\theta}+(1-\beta)(\alpha+k-1)]$$

and

$$T_{K,\alpha}^m = \left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right)^m.$$

Proof: Let $f(z)^\alpha \in S_{\lambda,l}^{m,\alpha}(A,B,b,\beta,\theta)$, then by the definition of subordination, we can express (4) as

$$1+\frac{1}{b}\left\{\frac{\beta e^{i\theta}\left(I^m(\lambda,l)f(z)^\alpha\right)' + z(1-\beta)\left[\left(I^m(\lambda,l)f(z)^\alpha\right)' + z\left(I^m(\lambda,l)f(z)^\alpha\right)''\right]}{\beta\left(I^m(\lambda,l)f(z)^\alpha\right)+(1-\beta)z\left(I^m(\lambda,l)f(z)^\alpha\right)'}-\alpha\right\} = \frac{1+Aw(z)}{1+Bw(z)}.$$

It implies that

$$\frac{\beta e^{i\theta}\left(I^m(\lambda,l)f(z)^\alpha\right)' + z(1-\beta)\left[\left(I^m(\lambda,l)f(z)^\alpha\right)' + z\left(I^m(\lambda,l)f(z)^\alpha\right)''\right]}{\beta\left(I^m(\lambda,l)f(z)^\alpha\right)+(1-\beta)z\left(I^m(\lambda,l)f(z)^\alpha\right)'}-\alpha$$

$$=\left[b(A-B)-B\left\{\frac{\beta e^{i\theta}\left(I^m(\lambda,l)f(z)^\alpha\right)' + z(1-\beta)\left[\left(I^m(\lambda,l)f(z)^\alpha\right)' + z\left(I^m(\lambda,l)f(z)^\alpha\right)''\right]}{\beta\left(I^m(\lambda,l)f(z)^\alpha\right)+(1-\beta)z\left(I^m(\lambda,l)f(z)^\alpha\right)'}-\alpha\right\}\right]w(z).$$ That is

$$\frac{\sum_{k=2}^{\infty}(\alpha+k-1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+k-1)\right]T_{k,\alpha}^{m}a_k(\alpha)z^{k-1}+\alpha\beta(e^{i\theta}-1)}{\left[\beta+\alpha(1-\beta)\right]+\sum_{k=2}^{\infty}\left[\beta+(1-\beta)(\alpha+k-1)\right]T_{k,\alpha}^{m}a_k(\alpha)z^{k-1}}$$

where

$$T_{K,\alpha}^{m}=\left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right)^{m}.$$

Thus, since $|w(z)|\le 1$, then

$$\left|\sum_{k=2}^{\infty}(\alpha+k-1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+k-1)\right]T_{k,\alpha}^{m}a_k(\alpha)z^{k-1}+\alpha\beta(e^{i\theta}-1)\right|$$

$$\le\left|b(A-B)\left[\beta+\alpha(1-\beta)\right]-B\,\alpha\beta(e^{i\theta}-1)-\sum_{k=2}^{\infty}\begin{bmatrix}B(\alpha+k-1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+k-1)\right]\\-b(A-B)\left[\beta+(1-\beta)(\alpha+k-1)\right]\end{bmatrix}T_{k,\alpha}^{m}a_k(\alpha)z^{k-1}\right|$$

and $T_{k,\alpha}^{m}$ is as earlier defined.

Letting $|z|\to 1^{-}$ through real values, we obtain

$$\frac{\sum_{k=2}^{\infty}\left[(\alpha+k-1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+k-1)\right]+\left|\begin{matrix}b(A-B)\left[\beta+(1-\beta)(\alpha+k-1)\right]\\-B(\alpha+k-1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+k-1)\right]\end{matrix}\right|\right]T_{k,\alpha}^{m}|a_k(\alpha)|}{|b|(A-B)\left[\beta+\alpha(1-\beta)\right]-\alpha\beta(e^{i\theta}-1)(B+1)}\le 1$$

which is the required result.

→ **Corollary 1:** Let f be a function of the form (2), then $f(z)\in S_{\lambda,l}^{m,\alpha}(A,B,b,\beta,0)$ if and only if

$$\frac{\sum_{k=2}^{\infty}\left[(\alpha+k-1)[\beta+(1-\beta)(\alpha+k-1)]+|M|\right]T_{K,\alpha}^{m}|a_k(\alpha)|}{|b|(A-B)\left[\beta+\alpha(1-\beta)\right]}\le 1$$

where

$$M=b(A-B)[\beta+(1-\beta)(\alpha+k-1)]-B(\alpha+k-1)[\beta+(1-\beta)(\alpha+k-1)]$$

and

$$T_{K,\alpha}^{m}=\left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right)^{m}.$$

→ **Corollary 2:** Let f be a function of the form (2), then $f(z)\in S_{\lambda,l}^{m,\alpha}(A,B,b,1,0)$ if and only if

$$\frac{\sum_{k=2}^{\infty}\left[(\alpha+k-1)+|M|\right]T_{K,\alpha}^{m}|a_k(\alpha)|}{|b|(A-B)}\le 1$$

where

$$M=b(A-B)-B(\alpha+k-1)$$

and

$$T_{K,\alpha}^{m}=\left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right)^{m}.$$

Growth and Distortion theorems

Theorem 2: If $f(z)\in S_{\lambda,l}^{m,\alpha}(A,B,b,\beta,\theta)$, then

$$r^{\alpha}-r^{\alpha+1}\left\{\frac{|b|(A-B)\left[\beta+\alpha(1-\beta)\right]-\alpha\beta(e^{i\theta}-1)(B+1)}{\left[(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]+\left|b(A-B)\left[\beta+(1-\beta)(\alpha+1)\right]-B(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]\right|\right]T_{2,\alpha}^{m}}\right\}$$

$$\leq$$

$$\left|f(z)^{\alpha}\right|$$

$$\leq$$

$$r^{\alpha}+r^{\alpha+1}\left\{\frac{|b|(A-B)\left[\beta+\alpha(1-\beta)\right]-\alpha\beta(e^{i\theta}-1)(B+1)}{\left[(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]+\left|b(A-B)\left[\beta+(1-\beta)(\alpha+1)\right]-B(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]\right|\right]T_{2,\alpha}^{m}}\right\}.$$

Equality is attained for function $f(z)^{\alpha}$ of the form

$$f(z)^{\alpha}=$$

$$z^{\alpha}+\left\{\frac{|b|(A-B)\left[\beta+\alpha(1-\beta)\right]-\alpha\beta(e^{i\theta}-1)(B+1)}{\left[(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]+\left|b(A-B)\left[\beta+(1-\beta)(\alpha+1)\right]-B(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]\right|\right]T_{2,\alpha}^{m}}\right\}z^{\alpha+1}.$$

Proof: From theorem 1, we have that

$$\sum_{k=2}^{\infty}\left|a_k(\alpha)\right|$$

$$\leq\frac{|b|(A-B)\left[\beta+\alpha(1-\beta)\right]-\alpha\beta(e^{i\theta}-1)(B+1)}{\left[(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]+\left|b(A-B)\left[\beta+(1-\beta)(\alpha+1)\right]-B(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]\right|\right]T_{2,\alpha}^{m}}. \qquad (5)$$

Using (2) and (5), we have that

$$\left|f(z)^{\alpha}\right|\leq|z|^{\alpha}+\sum_{k=2}^{\infty}\left|a_k(\alpha)\right||z|^{\alpha+k-1}\leq r^{\alpha}+r^{\alpha+1}\sum_{k=2}^{\infty}\left|a_k(\alpha)\right|.$$

That is

$$\left|f(z)^{\alpha}\right|$$

$$\leq r^{\alpha}+r^{\alpha+1}\left\{\frac{|b|(A-B)\left[\beta+\alpha(1-\beta)\right]-\alpha\beta(e^{i\theta}-1)(B+1)}{\left[(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]+\left|b(A-B)\left[\beta+(1-\beta)(\alpha+1)\right]-B(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]\right|\right]T_{2,\alpha}^{m}}\right\}.$$

Also,

$$\left|f(z)^{\alpha}\right|\geq|z|^{\alpha}-\sum_{k=2}^{\infty}\left|a_k(\alpha)\right||z|^{\alpha+k-1}\geq r^{\alpha}-r^{\alpha+1}\sum_{k=2}^{\infty}\left|a_k(\alpha)\right|.$$

That is

$$\left|f(z)^{\alpha}\right|$$

$$\geq r^{\alpha}-r^{\alpha+1}\left\{\frac{|b|(A-B)\left[\beta+\alpha(1-\beta)\right]-\alpha\beta(e^{i\theta}-1)(B+1)}{\left[(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]+\left|b(A-B)\left[\beta+(1-\beta)(\alpha+1)\right]-B(\alpha+1)\left[\beta e^{i\theta}+(1-\beta)(\alpha+1)\right]\right|\right]T_{2,\alpha}^{m}}\right\}.$$

Theorem 3: If $f(z)\in S_{\lambda,l}^{m,\alpha}(A,B,b,\beta,\theta)$, then

$$\alpha r^{\alpha-1} - r^{\alpha}\left\{\frac{(\alpha+1)\{|b|(A-B)[\beta+\alpha(1-\beta)]-\alpha\beta(e^{i\theta}-1)(B+1)\}}{\left[(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]+\left|b(A-B)[\beta+(1-\beta)(\alpha+1)]-B(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]\right|\right]T_{2,\alpha}^{m}}\right\}$$

$$\leq$$

$$\left|f'(z)^{\alpha}\right|$$

$$\leq$$

$$\alpha r^{\alpha-1} + r^{\alpha}\left\{\frac{(\alpha+1)\{|b|(A-B)[\beta+\alpha(1-\beta)]-\alpha\beta(e^{i\theta}-1)(B+1)\}}{\left[(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]+\left|b(A-B)[\beta+(1-\beta)(\alpha+1)]-B(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]\right|\right]T_{2,\alpha}^{m}}\right\}$$

Equality is attained for function $f(z)^{\alpha}$ of the form

$$f(z)^{\alpha} =$$

$$z^{\alpha} + \left\{\frac{|b|(A-B)[\beta+\alpha(1-\beta)]-\alpha\beta(e^{i\theta}-1)(B+1)}{\left[(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]+\left|b(A-B)[\beta+(1-\beta)(\alpha+1)]-B(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]\right|\right]T_{2,\alpha}^{m}}\right\}z^{\alpha+1}. \quad (6)$$

Proof: From (5), we can write that

$$\sum_{k=2}^{\infty}(\alpha+k-1)|a_k(\alpha)|$$

$$\leq \frac{(\alpha+1)\{|b|(A-B)[\beta+\alpha(1-\beta)]-\alpha\beta(e^{i\theta}-1)(B+1)\}}{\left[(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]+\left|b(A-B)[\beta+(1-\beta)(\alpha+1)]-B(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]\right|\right]T_{2,\alpha}^{m}}. \quad (5)$$

With the aid of (2) and (6), we have that

$$\left|f'(z)^{\alpha}\right| \leq \alpha|z|^{\alpha-1} + \sum_{k=2}^{\infty}(\alpha+k-1)|a_k(\alpha)||z|^{\alpha+k-2} \leq \alpha r^{\alpha-1} + r^{\alpha}\sum_{k=2}^{\infty}(\alpha+k-1)|a_k(\alpha)|$$

$$\leq \alpha r^{\alpha-1} + r^{\alpha}\left\{\frac{(\alpha+1)\{|b|(A-B)[\beta+\alpha(1-\beta)]-\alpha\beta(e^{i\theta}-1)(B+1)\}}{\left[(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]+\left|b(A-B)[\beta+(1-\beta)(\alpha+1)]-B(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]\right|\right]T_{2,\alpha}^{m}}\right\}.$$

Similarly,

$$\left|f'(z)^{\alpha}\right| \geq \alpha|z|^{\alpha-1} - \sum_{k=2}^{\infty}(\alpha+k-1)|a_k(\alpha)||z|^{\alpha+k-2} \geq \alpha r^{\alpha-1} - r^{\alpha}\sum_{k=2}^{\infty}(\alpha+k-1)|a_k(\alpha)|$$

$$\geq \alpha r^{\alpha-1} - r^{\alpha}\left\{\frac{(\alpha+1)\{|b|(A-B)[\beta+\alpha(1-\beta)]-\alpha\beta(e^{i\theta}-1)(B+1)\}}{\left[(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]+\left|b(A-B)[\beta+(1-\beta)(\alpha+1)]-B(\alpha+1)[\beta e^{i\theta}+(1-\beta)(\alpha+1)]\right|\right]T_{2,\alpha}^{m}}\right\}.$$ and

this ends the proof.

3. References

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