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Bipartite Representation of a Hypercube with Respect to the Norm of a Vertex

A. Anto Kinsley
Associate Professor, Department of Mathematics, St. Xavier's College, Palayamkottai, India
D. Princy
Research Scholar, Department of Mathematics, St. Xavier's College, Palayamkottai, India

Abstract:
Given a n-dimensional hypercube Q_n, we form a bipartite structure of Q_n with respect to norm of a vertex. The norm of a vertex is the number of ones in its binary representation. We prove some theorems which are useful to form the structure of a bipartite hypercube with respect to norm of a vertex. Also the circumference of Q_n is 2^n.

Keywords: *Hypercube, norm of a vertex, degree, equi norm-dist vertices, equi dist-norm vertices, circumference.*

1. Introduction

The n-dimensional hypercube or n-cube is a highly concurrent multiprocessor architecture consisting of 2^n nodes each connected to ***n*** neighbors. Such machines have been advocated as ideal ensemble architectures because of their powerful interconnection.

The n-dimensional hypercube Q_n is an undirected graph whose vertex set consists of all binary vectors of length n and two vertices are joined by an edge whenever their binary representation differ in a single position is defined in [3]. The structural properties of hypercubes are important in application level. Among other things, we propose a theoretical characterization of the bipartite hypercube as a graph using the norm of the vertex which is defined in [4].

Note that any hypercube is a bipartite graph with an even number of vertices.

1.1. Definition

The *distance* $d(u, v)$ between two vertices u and v in a graph G is the minimum length of a path joining them and if there exists no such path then $d(u, v) = \infty$.

1.2. Definition

The *hypercube* (*n*- cube) Q_n, is defined as the graph whose vertex set is the set of ordered *n*- tuples of 0s and 1s (i.e., in *n* positions) and where two vertices are adjacent if their ordered n- tuples differ in exactly one position. That is, if *uv* is an edge in Q_n, then *u* and *v* differ in exactly one position among the n-positions.

The number of vertices in Q_n is 2^n. The number of edges in Q_n is $(2^{n-1})\, n$.

1.3. Example

The 3-cube Q_3. The number of vertices in Q_3 is $2^3 = 8$. The number of edges in Q_n is $(2^{3-1})3 = 8$.

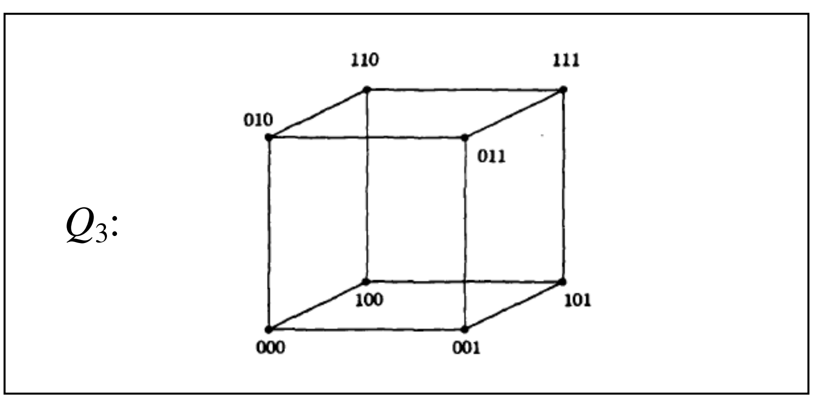

Figure 1: The graph Q_3.

The 3-D view of the hypercube shown in Figure 1 is defined in [5] also every hypercube Q_n is a regular bipartite graph with degree n for n ≥1 is proved in [5].

1.4. Remark
For any hypercube Q_n, $rad(G) = diam(G) = n$.

- Proof:

Let Q_n be a hypercube. Then every vertex v of Q_n is adjacent to n different vertices, which are different from v in a single position. Hence every vertex is farthest from a vertex such that the two vertices differ in n positions. Therefore the eccentricity of every vertex is n. Hence radius and diameter of every vertex is n. Thus every hypercube is a self-centered and self-periphery graph.

1.5. Definition
Norm of a vertex v belongs to $V(Q_n)$ is the number of 1s appears in the binary representation of v and is denoted by $\|v\|$.

1.6. Definition
The *circumference* of a graph G is the length of the longest cycle of G.

2. Construction of a Bipartite Hypercube with Respect to Norm of a Vertex
Bipartite hypercubes are already studied in [2]. We develop these bipartite concepts with norm of a vertex. Large hypercubes are such difficult to form, but using the norm of the vertices we can construct the large n-regular bipartite hypercubes.
Every hypercube has 2^n vertices. We divide the total number of vertices into two partitions, each partition has 2^{n-1} vertices. One partition has vertices of odd norm and the other partition has the vertices of even norm. In this paper we will refer to the hypercube or n-cube graph, or hypercube or n-cube, as the graph thus defined.
The graph Q_3 in Figure 1 is transformed into

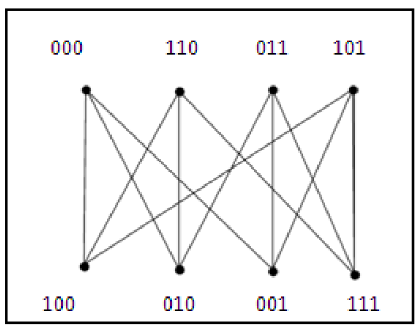

Figure 2: Bipartite view of a hypercube

2.1. Theorem
Let Q_n be a bipartite hypercube. Then

$$d(v_i, v_j) = \begin{cases} 1,3, \dots n-1 & \text{if } v_i \in B_1(or\ B_2) and\ v_j \in B_2(or\ B_1) \\ 2,4, \dots, n & \text{if } v_i, v_j \in B_1(or\ B_2) \end{cases} \text{ for } n \text{ is even and}$$

$$d(v_i, v_j) = \begin{cases} 1,3, \dots n & \text{if } v_i \in B_1(or\ B_2) and\ v_j \in B_2(or\ B_1) \\ 2,4, \dots, n-1 & \text{if } v_i, v_j \in B_1(or\ B_2) \end{cases} \text{ for } n \text{ is odd.}$$

- Proof:

Let Q_n be an n-regular bipartite graph. For n is even, let v_i belongs to B_1(or B_2) be a vertex. Then v_i is adjacent to n vertices in B_2(or B_1). Take a vertex $v_1 \in B_2$(or B_1) from these n vertices, we get $d(v_i, v_1) = 1$.
Then v_1 is adjacent to n-1 vertices other than v_i in B_1(or B_2). Take a vertex $v_2 \in B_1$(or B_2) from these above n-1 vertices. Then $d(v_i,v_2) = 2$, $d(v_i,v_3) = 3$, ….., $d(v_i,v_{n-1}) = n-1$.
Proceeding like this we get the following,

$$d(v_i, v_j) = \begin{cases} 1,3, \dots n-1 & \text{if } v_i \in B_1(or\ B_2) and\ v_j \in B_2(or\ B_1) \\ 2,4, \dots, n & \text{if } v_i, v_j \in B_1(or\ B_2) \end{cases}$$

Similarly, we proceed as above we get the following for n is odd

$$d(v_i, v_j) = \begin{cases} 1,3, \dots n & \text{if } v_i \in B_1(or\ B_2) and\ v_j \in B_2(or\ B_1) \\ 2,4, \dots, n-1 & \text{if } v_i, v_j \in B_1(or\ B_2) \end{cases}$$

2.2. Remark
In an hypercube Q_n number of vertices of odd norm and number of vertices of even norm is same.

2.3. Theorem

Let v_i, v_j be two vertices in the n-regular bipartite hypercube Q_n. Then $d(v_i, v_j) = n$ if and only if v_i and v_j are differ in n positions.

- Proof:

Let Q_n be an n-regular bipartite graph.

2.3.1. Case

Take n is odd. Assume that $d(v_i,v_j) = n$. Then $v_i \in B_1$(or B_2) and $v_j \in B_2$(or B_1).

First we take the vertex v_i belongs to B_1(or B_2). Then v_i is adjacent to n vertices in B_2(or B_1) such that v_i and these n vertices differ in one position. Take a vertex v_1 which belongs to B_2(or B_1) from these above n vertices we get $d(v_i, v_1) = 1$. Then v_1 which belongs to B_2(or B_1) is adjacent to n-1 vertices in B_1(or B_2) other than v_i such that v_1 and these n-1 vertices differ in one position. Take a vertex v_2 which belongs to B_1(or B_2) from these above n-1 vertices we get $d(v_1,v_2) = 1$ and $d(v_i,v_2) = 2$. Then v_i and v_2 are differ in two positions. Also v_2 is adjacent to n-1 vertices in B_2(or B_1) other than v_1 such that v_2 and these n-1 vertices differ in one position. Take a vertex v_3 belongs to B_2(or B_1) from the above n-1 vertices, we get $d(v_2,v_3) = 1$ and $d(v_i,v_3) = 3$. Then v_i and v_3 differ in three positions.

Similarly, we get $d(v_3,v_4) = 1$ and $d(v_i,v_4) = 4$ where v_4 belongs B_1(or B_2) and , v_i and v_4 differ in 4 positions. Also we have $d(v_4,v_5) = 1$ and $d(v_i,v_5) = 5$ where v_5 belongs to B_2(or B_1) and v_i and v_5 differ in five positions. Since n is odd, we get $d(v_{n-2},v_{n-1}) = 1$ and $d(v_i,v_{n-1}) = n$-1 where v_{n-1} belongs to B_1(or B_2) and v_i and v_{n-1} differ in n-1 positions. Hence v_i and v_j are differ in n positions.

2.3.2. Case

Take n is even. Assume that $d(v_i,v_j) = n$. Then v_i,v_j belongs to B_1(or B_2).

First we take the vertex v_i belongs to B_1(or B_2). Then v_i beongs to B_1(or B_2) is adjacent to n vertices in B_2(or B_1) such that v_i and these n vertices differ in one position. Take a vertex v_1 belongs to B_2(or B_1) from these above n vertices we get $d(v_i,v_j) = 1$.

Then v_1 belongs to B_2(or B_1) is adjacent to n-1 vertices in B_1(or B_2) other than v_i such that v_1 and these n-1 vertices differ in one position. As the above case,

$d(v_3,v_4) = 1$ and $d(v_i,v_4) = 4$ where v_4 belongs B_1(or B_2) and v_i and v_4 differ in four positions.

$d(v_4,v_5) = 1$ and $d(v_i,v_5) = 5$ where v_5 belongs to B_2(or B_1) and v_i and v_5 differ in five positions.

Since n is even, we get $d(v_{n-2},v_{n-1}) = 1$ and $d(v_i,v_{n-1}) = n$-1 where v_{n-1} belongs to B_1(or B_2) and v_i and v_{n-1} differ in n-1 positions. Hence v_i and v_j are differ in n positions.

Conversely,

Assume that v_i and v_j differ in n positions.

Take v_i belongs to B_1(or B_2). Then v_i is adjacent to n vertices in B_2(or B_1) differ in one position.

Therefore $d(v_i, v_1) = 1$, where v_1 is any vertex in B_2(or B_1) in the above n vertices.

Take a vertex v_2 belongs B_1(or B_2). Then $d(v_1, v_2) = 1$, $d(v_2, v_3) = 1$,……., $d(v_{n-1}, v_j) = 1$.

If n is odd, then v_j belongs to B_1(or B_2).

If n is even, then v_j belongs to B_2(or B_1).

We have $d(v_i, v_1) + d(v_1, v_2) + d(v_2, v_3) + \ldots + d(v_{n-1}, v_j) = 1+1+1+ \ldots +1$($n$ times).

Hence $d(v_i, v_j) = n$.

2.4. Theorem

Let Q_n be an hypercube. Then $\|v_i\|+\|v_j\|= n$ if $d(v_i, v_j) = n$. But the converse need not be true.

- Proof:

Assume that $d(v_i, v_j)$ = n. Then we have from the above theorem v_i and v_j differ in n positions whenever n is odd or n is even. Hence $\|v_i\|+\|v_j\|= n$.

2.5. Definition

Let v_i ,v_j be any two vertices in a bipartite hypercube. Then v_i and v_j are called *equi dist-norm* vertices to each other if $d(v_i, v_j) = k$ implies that $\|v_i\|+\|v_j\|= k$. The ordered pair (v_i, v_j) is called the *equi dist-norm pair.*

2.6. Definition

Let v_i ,v_j be any two vertices in a bipartite hypercube. Then v_i and v_j are called *equi norm-dist* vertices to each other if $\|v_i\|+\|v_j\|= k$ implies that $d(v_i, v_j) = k$. The ordered pair (v_i, v_j) is called the *equi norm-dist pair*.

2.7. Remark

If $d(v_i, v_j) = n$, then the pair (v_i, v_j) is an equi dist-norm pair if and only if (v_i, v_j) is an equi norm-dist pair.

2.8. Theorem

For a bipartite hypercube Q_n, the total number of equi dist-norm pair is 2^{n-1}.

- Proof:

We shall prove this theorem by induction on ‘n’. Take n =1 $d(0, 1) = 1$ implies $\|0\|+\|1\| = 1$. Similarly for n =2, $d(00, 11) = 2$ implies $\|00\|+\|11\|=2$ and $d(01, 10) = 2$ implies $\|01\|+\|10\|=2$.

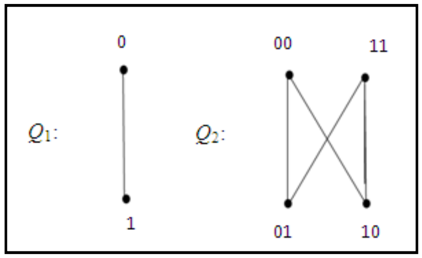

Figure 3: The cubes

Hence the result is true for n=1 and n=2. That is there is only one pair of equi dist-norm vertices in Q_1 and there are two pairs of equi dist-norm vertices in Q_2.
Now we assume that the theorem is true for n-1. That is the total number of pair of equi dist-norm vertices in Q_{n-1} are 2^{n-2}. We have to show that this theorem for Q_n.
In Q_{n-1}, there are 2^{n-1} vertices and the number of pair of equi dist-norm vertices in Q_{n-1} are 2^{n-2}. In Q_{n-1}, there are $2(2^{n-1}) = 2^n$ vertices and hence the number of pair of equi dist-norm vertices in Q_n are $2(2^{n-2}) = 2^{n-1}$. The theorem has been completed.

2.9. Theorem

The circumference of the bipartite hypercube Q_n is 2^n, for $n \geq 2$.

- Proof:

Let Q_n be the n-regular bipartite hypercube. Then $|Q_n| = 2^n$.
Hence there are cycles of even length. We shall prove this theorem by induction on ‘n’.
Take n = 2. Then $|Q_2| = 2^2 = 4$ and $V(Q_2) = \{00,01,10,11\}$,
$E(Q_2) = \{(00)(01),(01)(11),(10)(11),(00)(11)\}$. Hence the circumference of Q_2 is $2^2 = 4$.
Assume that this theorem is true for n-1. Hence circumference of Q_{n-1} is 2^{n-1}.
That is each partition has 2^{n-2} elements.
Add each vertex to a partition and make the adjacency with respect to the difference in one position. Hence we get a graph with 2^n vertices and of circumference 2^n. Hence circumference of Q_n is 2^n.

2.10. Remark

There are no cycle of odd length in an hypercube. This is already proved in [5]. Let Q_n be a n-regular bipartite hypercube, where n 2. Then the induced cyles of Q_n are of length 4, 6, 8,….., 2^{n-1}.
The circumference of a hypercube is a Hamiltonian cycle of length 2^{n-1}. Hence every hypercube is a Hamiltonian graph. Thus the Hamiltonian path of any hypercube is of length 2^{n-1}-1. Also the girth of Q_n is 4.

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