THE INTERNATIONAL JOURNAL OF SCIENCE & TECHNOLEDGE

Fixed Point Theorem in Banach Space

Dr. Ganesh Kumar Soni
Head, Department of Mathematics, Govt. P.G. College Narsinghpur, Madhya Pradesh, India

Abstract:
The purpose of this paper is to obtain a common fixed theorem in Banach space using a contractive type condition Work on common fixed point have done by many authors such as Iseki K., Kannan R. , Rus I.A., Sehgal S.L , Yen C.L , Singh S.L , Fisher B. , Rhoades B.E. and Sessa S., Ray, B.K. and Chatterjee H. ,Sharma P.L. and Rajput ,S.S, Sharma P.L. and Bajaj N.;, etc. have also established interesting results on common fixed point. We prove let(X,d) be a Banach space. Let P , S and T are three self mapping as P, S and T : X → X satisfy in the conditions

$$||SPx - TPy|| \le \alpha \frac{||x-SPx||\,||x-TPy||}{||x-TPy|| + ||x-SPx|| + ||x-y||} + \beta \frac{||x-y||[1+\sqrt{||x-y||\,||x-TPy||} + \sqrt{||x-y||\,||y-TPy||}\,]}{[1+||x-y|| + ||x-SPx||\,||x-TPy||\,||y-SPx||\,||y-TPy||]}$$

$$+\gamma \frac{||x-SPx||[1+\sqrt{||x-TPy|| + ||y-SPx||}\,]}{[1+\sqrt{||x-SPx|| + ||y-TPy||}\,]}$$

for all x,y in X and α , β , γ are reals and α+β+γ < 1. Futher assume that either SP=PS or TP = PT.Then S,T and P have a common unique fixed point in X.

1. Introduction

Work on common fixed point have done Many authors by Iseki [1] Kannan [2] Rus [3] Sehgal [4] Iseki [5]Yen [6] Sing [7] Fisher [8] Rhoades and Sessa [9] Rayand Chatterjee [10] Sharma and Rajput [11]Sharma and Bajaj [12] etc. Have also established interesting results on common fixed point.

2. Our Main Result

→ Theorem: -(X, d) be a Banach space.Let P,S and T are three self-mapping asP, S and T: X → X satisfy in the following conditions:

$$||SPx-TPy|| \le \alpha \frac{||x-SPx||\,||x-TPy||}{||x-TPy|| + ||x-SPx|| + ||x-y||}$$
$$+\beta \frac{||x-y||[1+\sqrt{||x-y||\,||x-TPy||} + \sqrt{||x-y||\,||y-TPy||}]}{[1+||x-y|| + ||x-SPx||\,||x-TPy||\,||y-SPx||\,||y-TPy||]}$$
$$+\gamma \frac{||x-SPx||[1+\sqrt{||x-TPy|| + ||y-SPx||}]}{[1+\sqrt{||x-SPx|| + ||y-TPy||}]} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

for all x, yin X and α,β, γ are realsand α+β+γ < 1. Further assume that either SP=PS orTP = PT. ThenS, TandP have a common unique fixed point in X.

→ Proof: -Let $x_0 \in X$ such that$x_{2n+1} = SPx_{2n}$ and $x_{2n} = TPx_{2n-1}$now using inequality then we have

$||SPx_{2n} - TPx_{2n-1}|| = ||x_{2n+1} - x_{2n}||$

$$\le \alpha \frac{||x_{2n}-SPx_{2n}||\,||x_{2n}-TP\,x_{2n-1}||}{||x_{2n}-TPx_{2n-1}|| + ||x_{2n}-SPx_{2n}|| + ||x_{2n}-x_{2n-1}||}$$
$$+\beta \frac{||x_{2n}-x_{2n-1}||\left[1+\sqrt{||x_{2n}-x_{2n-1}||\,||x_{2n}-TPx_{2n-1}||} + \sqrt{||x_{2n}-x_{2n-1}||\,||x_{2n-1}-TPx_{2n-1}||}\right]}{[1+||x_{2n}-x_{2n-1}|| + ||x_{2n}-SPx_{2n}||\,||x_{2n}-TPx_{2n-1}||\,||x_{2n-1}-SPx_{2n}||\,||x_{2n-1}-TPx_{2n-1}||]}$$
$$+\gamma \frac{||x_{2n}-SPx_{2n}||\left[1+\sqrt{||x_{2n}-TPx_{2n-1}|| + ||x_{2n-1}-SPx_{2n}||}\right]}{\left[1+\sqrt{||x_{2n}-SPx_{2n}|| + ||x_{2n-1}-TPx_{2n-1}||}\right]}$$

$$\le \alpha \frac{||x_{2n}-x_{2n+1}||\,||x_{2n}-x_{2n}||}{||x_{2n}-x_{2n}|| + ||x_{2n}-x_{2n+1}|| + ||x_{2n}-x_{2n-1}||}$$

$$+\beta\frac{||x_{2n}-x_{2n-1}||\left[1+\sqrt{||x_{2n}-x_{2n-1}||\,||x_{2n}-x_{2n}||}+\sqrt{||x_{2n}-x_{2n-1}||\;||x_{2n-1}-x_{2n}||}\right]}{[1+||x_{2n}-x_{2n-1}||+||x_{2n}-x_{2n+1}||\,||x_{2n}-x_{2n}||\,||x_{2n-1}-x_{2n+1}||\,||x_{2n-1}-x_{2n}||]}$$

$$+\gamma\frac{||x_{2n+1}-x_{2n}||\left[1+\sqrt{||x_{2n}-x_{2n}||+||x_{2n-1}-x_{2n+1}||}\right]}{\left[1+\sqrt{||x_{2n}-x_{2n+1}||+||x_{2n-1}-x_{2n}||}\right]}$$

$$\leq \beta\,||x_{2n}-x_{2n-1}||+\gamma\,||x_{2n+1}-x_{2n}||$$

$$||x_{2n+1}-x_{2n}||\leq \frac{\beta}{(1-\gamma)}||x_{2n}-x_{2n-1}||$$

$$||x_{2n+1}-x_{2n}||\leq h||x_{2n}-x_{2n-1}||$$

where $0\leq h=(\beta+\gamma)<1$

Continuing in this way, we have $||x_{2n+1}-x_{2n}||\leq h^{2n}||x_0-x_1||$

Similarly,we can show that

$$||x_{2n+1}-x_{2n+2}||\leq h^{2n+1}||x_0-x_1||$$

It can we easily seen that $\{x_n\}$ is a Cauchy sequence in X and X is the Banach Space. So there exists z Ɛ X such

that $x_n\to z$ as $n\to\infty$ then from (1) we have

$$||SPz-x_{2n}||=||SPz-TPx_{2n-1}||\leq\alpha\frac{||z-SPz||\;||z-x_{2n}||}{||z-x_{2n}||+||z-SPz||+||z-x_{2n-1}||}$$

$$+\beta\frac{||z-x_{2n-1}||\left[1+\sqrt{||z-x_{2n-1}||\,||z-x_{2n}||}+\sqrt{||z-x_{2n-1}||\;||x_{2n-1}-x_{2n}||}\right]}{[1+||z-x_{2n-1}||+||z-SPz||\,||z-x_{2n}||\,||x_{2n-1}-SPz||\,||x_{2n-1}-x_{2n}||]}$$

$$+\gamma\frac{||z-SPz||\left[1+\sqrt{||z-x_{2n}||+||x_{2n-1}-SPz||}\right]}{\left[1+\sqrt{||z-SPz||+||x_{2n-1}-x_{2n}||}\right]}$$

Letting $n\to\infty$ we get that

\ $||SPz-z||\leq\gamma||z-SPz||$

It follows that SPz = zas $\gamma<1$.

Similarly, by condersing $||x_{2n+1}-TPz||$we have

SPz=TPz=z ……………………………………….(2)

Let SP=PS then

$$||Pz-z||=||PSPz-TPz||=||SPPz-TPz||\leq\alpha\frac{||Pz-SPPz||\,||Pz-TPz||}{||Pz-TPz||+||z-SPPz||+||Pz-z||}$$

$$+\beta\frac{||Pz-z||\left[1+\sqrt{||Pz-z||\,||Pz-TPz||}+\sqrt{||Pz-z||\,||z-TPz||}\right]}{[1+||Pz-z||+||Pz-SPPz||\,||Pz-TPz||\;||z-SPPz||\,||z-TPz||]}$$

$$+\gamma\frac{||Pz-SPPz||\left[1+\sqrt{||Pz-TPz||+||z-SPPz||}\right]}{\left[1+\sqrt{||Pz-SPPz||+||z-TPz||}\right]}$$

$$\leq\alpha\frac{||Pz-Pz||\,||Pz-z||}{||Pz-z||+||z-Pz||+||Pz-z||}$$

$$+\beta\frac{||Pz-z||\left[1+\sqrt{||Pz-z||\,||Pz-z||}+\sqrt{||Pz-z||\,||z-z||}\right]}{[1+||Pz-z||+||Pz-Pz||\,||Pz-z||\;||z-Pz||\,||z-z||]}$$

$$+\gamma\frac{||Pz-Pz||\left[1+\sqrt{||Pz-z||+||z-Pz||}\right]}{\left[1+\sqrt{||Pz-Pz||+||z-z||}\right]}$$

$$\leq\beta||Pz-z||$$

Since $\beta<1$ it follows that Pz=z. Now from (2) Pz = z = Tz.

Similarly using conditions if PT=TP we obtainPz=z=Sz=Tz. The uniqueness of common fixed point can we proved easily by (1.) Thus the theorem is proved.

3. References

i. Iseki ,K.;"Fixed point theorems in Banach spaces" Math. Sem. Notes Kobe Univ.Vol.2No1paperNo.3(1974)4pp.
ii. Kannan, R, :"Some results on fixed points II" Amer. Math" Monthly 76(1969), 405-408.
iii. Rus, I.A.: "On common fixed point" Studia Univ. Babes-Bolyai 18(1973) 31-33.
iv. Seghal, V.M.: "On fixed point and coment fixed point theorems in metric space" Canad. Math Bull. 17(2) (1974) 257-259.
v. Iseki, K. : "Common fixed point of mappings" Tamkang Jour, Mathe. 6(2)(1975)157-159.
vi. Yen, C.L. :" Remark on common fixed points" Tamkang J, Math. 3(1972) 95-96.
vii. Singh, S.L. : "On common fixed point of commuting mappings"Math sem. Notes Kobe Univ. 5(1977) 113-114.
viii. Fisher, B." Common fixed points of commuting mapping" Bull. Inst. Math. Acad. Sincia 9(1981) 399-406.
ix. Rhoades, B.E. and Sessa, S. : "Common fixed point theorems for three mappings under a weak commutativity condition" Indian J. Pure appl. Math. 17(1)(1986) 47-57.
x. Ray,B.K. and Chatterjee, H.;"On some results on fixed points in metric and Banach spaces "Indain J. Pure appl. Math.Vol.8(8) (1977)955-960
xi. Sharma, P. L. and Rajput,S.S.;"Fixed point theorems in Banach spaces" Vikram Mathematical Journal Vol.4(1983)35-38.
xii. Sharma,P.L.and Bajaj,N.;"Fixed point theorems in Banach spaces for commuting mappings" Jornal of M.A.C.T.16(1983).