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Semi-pre Door Spaces

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Abstract:
In this paper we introduce sp- door space with the aid of semi-preopen sets defined by Andrijević [1]. Some basic properties of sp-door space including invariance is studied in this paper.
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1. Introduction

The concept of door space was studied in some details by J. L. Kelley [6]. J. Dontchev [4] carried out further investigation on door space. In 1968, J.P. Thomas [10] defined semi-door space using semi open sets of Levine [7]. We introduce semi-pre door space (briefly sp-door space) utilising semi-preopen sets of Andrijević [1]. In section 2 of this paper some known definitions and result necessary for the presentation of the paper are given.

2. Preliminaries

Throughout the paper (X, τ) or X always denotes a non trivial topological space. The family of all open sets containing x is denoted by Σ(x). Interior and closure of a subset A of X is denoted by Int(A) and Cl(A) respectively.

- → Definition 2.1. $A \subset X$ is called a semi-preopen [1] (resp. semi-open [7], pre-open [8], α-open [9]) set briefly s.p.o. set if $A \subset Cl(Int(Cl(A)))$ (resp. $A \subset Cl(Int(A))$, $A \subset Int(Cl(A))$, $A \subset Int(Cl(Int(A)))$). The family of all semi-preopen (resp.pre-open) sets of X is denoted by SPO(X) (resp.PO(X)). For each $x \in X$, the family of all s.p.o. sets containing x is denoted by SPO(X, x).
- → Definition 2.2. The complement of a s.p.o.(resp.s.o.) set is called semi-preclosed[1] (resp. semi-closed [7]).
- → The family of all semi-preclosed(resp.semi-closed) sets set of X is denoted by SPF(X) (resp.SC(X)) .
- → Definition 2.3. The semi-preclosure [1] of $A \subset X$ is denoted by spcl (A) and is defined by spcl (A) = $\cap\{B : B$ is semi-preclosed and $B \supset A\}$.
- → Definition 2.4. A topological space X is said to be sp- T_2 [5] iff for every pair of distinct points $x,y\in X$ there exist disjoint sets $U\in SPO(X,x)$ and $V\in SPO(X,y)$.
- → Definition 2.4. A topological space X is said to be
  - i) submaximal [2] if every dense subset of X is open;
  - ii) irreducible [3] if every open subset of X is dense;
  - iii) door space [6] iff for every $A \subset X$ either A is open or closed;
  - iv) semi-door space [10] iff for every $A \subset X$ either $A \in SO(X)$ or $A \in SC(X)$.
- → Lemma 2.1 [5]. In a topological space X if $A \in SPO(X)$ and B is an α- set , then $A \cap B \in SPO(X)$.
- → Lemma 2.2 [5]. Let $A \subset Y \subset X$ and $Y \in PO(X)$, then $A \in SPO(X)$ iff $A \in SPO(Y)$.

3. We Begin with the Following Definition

- → Definition 3.1. A topological space is called semi-pre door space (briefly sp-door space) if for every subset A of X, either $A \in SPO(X)$ or $A \in SPF(X)$.
- → Remark 3.1. Clearly a door space is a sp-door space but not conversely as the following example shows.
- → Example 3.1. Let X = {a, b, c} be the set with the topology τ = {ϕ, X, {a}}.Then SPO (X) = {ϕ, X, {a}, {a, b}, {a, c}}.Clearly X is sp-door space but not a door space.
- → Remark 3.2. Evidently every semi-door space is a sp-door space but the converse is not always true as is exhibited in the next example.
- → Example 3.2. Let X= {a,b,c} be endowed with the topology τ = {ϕ, X, {a, b}}. Then SO(X) = τ , SPO(X)= { ϕ, X, {a},{b},{a,b },{a,c},{b,c}}.Clearly X is a sp-door space but not a semi-door space.

To examine the various properties of a sp-door space we need the following definitions and lemmas.

→ Definition 3.2. Let $x \in X$ and $A \subset X$. Then x is said to be a semi-pre limit point (briefly sp-limit point) iff $A \cap (U - \{x\}) \neq \phi$ for all $U \in SPO(X, x)$.

→ Remark 3. 3. Every sp-limit point of a set A is a limit point of A. But the converse is not true.

→ Example 3.3. Let (X, τ) be the space of example 3.2.Take $A = \{a, c\}$. Then b is a limit point of A but not a sp-limit point of A.

→ Definition 3.3 A subset A of X is said to be semi-predense (briefly sp-dense) iff spcl (A) = X.

→ Remark 3.4. Obviously every sp-dense set is dense but a dense subset may not be sp-dense as is clear from the following example.

→ Example 3.4. Let (X, τ) be the space of Example 3.2 .Take $A = \{a\}$. Then A is dense but not sp-dense.

→ Definition 3.4. A space X is said to be semi-pre submaximal (briefly sp-submaximal) iff every sp-dense subset of X is s.p.o.

→ Remark 3.5. Clearly every submaximal space is sp-submaximal but the converse need not be true. This is clear from the next example.

→ Example 3.5. Let (X, τ) be the space of Example 3.1. Taking $A = \{a, b\}$ it is easy to see that Cl (A) = X but $A \notin \tau$. Thus (X, τ) is not submaximal. On the other hand the sp-dense subsets of X are $\{a\}$, $\{a, b\}$, $\{a, c\}$ and X. All these sets are s.p.o. sets and hence (X, τ) is sp-submaximal.

→ Definition 3.5. A space X is said to be semi-pre irreducible (briefly sp-irreducible) iff every semi-preopen subset of X is sp-dense.

→ Remark 3.6. Evidently every sp-irreducible space is irreducible but that the converse may not be true can be seen from the next example.

→ Example 3.6. Let $X = \{a, b\}$ be the set with the indiscrete topology τ. Then (X, τ) is irreducible. That (X, τ) is not sp-irreducible follows from the fact that $\{a\} \in SPO(X)$, spcl $(\{a\}) = \{a\}$ but $\{a\}$ is not sp-dense.

→ Lemma 3.1. In a topological space X, if for a point $x \in X$, $\{x\} \in SPO(X)$, then x is not a sp-limit point of X.

The straightforward proof is omitted.

→ Theorem 3.1. If X is a Hausdorff sp-door space, then X has at most one sp-limit point.

➢ Proof. Let us assume that $x, y \in X$ $(x \neq y)$ be two sp-limit points of X. Hausdorffness of X provides two disjoint open sets $U \in \Sigma(x)$ and $V \in \Sigma(y)$. Set $A = (U - \{x\}) \cup \{y\}$. Since X is a sp-door space either $A \in SPO(X)$ or $A \in SPF(X)$. Now if $A \in SPO(X)$. then by Lemma 2.1 and the fact that every open set is an α-set it follows that $A \cap V \in SPO(X)$. But $A \cap V = \{y\} \Rightarrow \{y\} \in SPO(X)$. Again if $A \in SPF(X)$ then $X - A \in SPO(X) \Rightarrow (X - A) \cap U = ((U - \{x\}) \cup \{y\})^C \cap U$. An application of De Morgan's law produces $(X - A) \cap U = ([U^C \cup \{x\}] \cap \{y\}^C) \cap U = \phi \cup \{x\} = \{x\}$. So from above, we get $\{x\} = (X - A) \cap U \in SPO(X)$. Thus one of $\{x\}$ and $\{y\}$ must be a s.p.o. set and therefore by Lemma 3.1 least one of x and y is not a sp-limit point. This contradicts our assumption. Hence the theorem.

→ Theorem 3.2. If X is a sp-T_2 door space then X has at most one sp-limit point.

→ Proof. Pursuing the same reasoning with minor modification in the proof of Theorem 3.1 in the result follows.

→ Lemma 3.2. For any $A \subset X$, $spcl_Y(A) = spcl_X(A) \cap Y$.

Proof involves same argument as in classical case and therefore leftout.

→ Theorem 3.3. Every α-subspace of a sp-door space is a sp-door space.

➢ Proof. Let X be a sp-door space ,Y be an α-set and $A \subset Y$. Since X is a sp-door space either $A \in SPO(X)$ or $A \in SPF(X)$. Now if $A \in SPO(X)$ then since every α-set is a p.o. set, by Lemma 2.2 one obtains $A \in SPO(Y)$. Again if Let $A \in SPF(X)$ then by Lemma 3.2 it follows that $spcl_Y(A) = spcl_X(A) \cap Y = A \cap Y = A \Rightarrow A \in SPF(Y)$. Hence Y is a sp-door space.

→ Theorem 3.4. Every sp-door space is sp-submaximal.

➢ Proof. Let $A \subset X$ be sp-dense. Now sp-doorness of X indicates that either $A \in SPO(X)$ or $A \in SPF(X)$. If $A \in SPO(X)$, then there is nothing to prove. If $A \in SPF(X)$, then spcl (A) = A. But sp-denseness of X indicates that spcl(A) = X ,whence from above, one obtains $A = X \Rightarrow A \in SPO(X)$. Hence A is a sp-door space.

→ Theorem 3.5. Every sp-irreducible and sp-submaximal space is a sp-door space.

➢ Proof. Let X be a sp-irreducible and sp-submiximal space and $A \subset X$. Now if A is sp-dense then sp-submaximality of X guarantees that $A \in SPO(X)$. If A is not sp-dense then spcl $(A) \neq X$. Let $x \in X$ be such that $x \notin$ spcl (A) .Then there exists a $U \in SPO(X, x)$ such that $U \cap A = \phi \Rightarrow U \subset X - A$.The sp-irreducibility of X yields that U is sp-dense. So, X = spcl (U).One now infers from above that spcl $(X - A) = X$ ie. $X - A$ is also sp-dense. Again sp-submaximality of X implies that $X - A \in SPO(X)$ which indicates that $\Rightarrow A \in SPF(X)$.Therefore in any case either $A \in SPO(X)$ or $A \in SPF(X)$. Hence X is a sp-door space.

4. Transfer Topology of *sp*-Door Space

→ Definition 3.6 . A mapping $f : X \to Y$ is said to be sp-open if $f[A] \in SPO(Y)$ whenever $A \in SPO(X)$.

→ Theorem 3.6. Let $f : X \to Y$ be sp-open and surjective. If X is a sp-door space, then Y is a sp-door space.

- Proof. Let $A \subset Y$. Since X is a sp-door space either $f^{-1}[A] \in$ SPO (X) or $f^{-1}[A] \in$ SPF (X). If $f^{-1}[A] \in$ SPO (X) then the surjectivity and sp-openness of f together yield that $A = f f^{-1}[A] \in$ SPO (Y). Again if $f^{-1}[A] \in$ SPF (X) then $X - f^{-1}[A] = f^{-1}[Y - A] \in$ SPO (X).Pursuing the same technique we may see that $Y - A \in$ SPO (Y) $\Rightarrow A \in$ SPF (Y).Thus either $A \in$ SPO (Y) or $A \in$ SPF (Y). In otherwords, Y is a sp-door space.

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