THE INTERNATIONAL JOURNAL OF
SCIENCE & TECHNOLEDGE

Basis and Domination in Disemigraph

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Abstract:
The notion of basis introduced by Dénes König [8] as point-basis, is a well-studied topic in the theory of digraph having significant role both from applied and theoretical point of view.
Domination in Graph Theory, on the other hand is another important topic, historically claimed to have originated from the game of chess. However, the concrete notion of domination in graph found its formal shape in the works of C. Berge [6] and Ore in [22].
In this paper, attempt has been made to initiate a study of notions of basis and domination along with some characterizations of relationships between the two notions.

Keywords: *Semigraph, disemigraph, basis, domination.*
AMS 2010subject classification no.: 05C20, 05C69 and 05C99.

1. Introduction

The notion of semigraph is the outcome of an attempt by E. Sampath kumar [2] to generalize the notion of a graph. Semigraph is thus a generalization of graph, which resembles graph when drawn in a plane and where most of the notions or results of graph can find their natural counterparts in a generalized setting. The beauty of semigraph lies in the variety of definitions, concepts and results, most of which coincide with corresponding ideas for graphs.
It is well-known that graph theory is related not only to areas of pure and applied mathematics, but many diverse fields of knowledge also. Graph theory is a study of some 2-tuples (u,v) of distinct elements belonging to a set V, while the Semigraph theory is a study of n-tuples $(u_1,u_2,\ldots\ldots,u_n)$ of distinct elements belonging to a set V, for various $n\geq2$, with the conditions:
i) any n-tupple $(u_1, u_2,\ldots\ldots,u_n)$ is the same as the n-tupple $(u_n,u_{n-1},\ldots\ldots,u_1)$ and
ii) any two such n-tuples have at most one element in common.
Since an edge in a semigraph geometrically resembles an edge in a graph, one can easily give a direction or an orientation to each edge and obtain a structure called directed semigraph analogous to the concept of directed graph.
There is another significant generalization of graph, called Hypergraph, which is due to C. Berge [7].A hypergraph H (V, E) with a vertex set V and an edge set E differs from a graphin that, in a hypergraph an edge can connect more than two vertices, that is, an edge is an arbitrary subset of the vertex set V. While both hypergraph and semigraph allow edges with more than two vertices, however, the vertices in any edge of a semigraph follow a particular order though the vertices in an edge of a hypergraph do not have any such order. The particular way of arranging the vertices of an edge in a semigraph makes the semigraph more akin to graph than hypergraph.
Basis, domination and other related concepts are extensively studied for graphs and hypergraphs.However, corresponding analogues of these notions in semigraph or disemigraph have not received much attention yet.
Point-basis in digraph was introduced by Dénes König [8].König characterized point-basis in a weak sense and Harary et. al. [21] gave some sort of its stronger characterization for finite digraphs. E. Sampathkumar introduced the concept of cycle basis in [2]. No further development is found in the literature regarding basis in semigraphs or disemigraphs.
Historically, the first domination-type problems originated from the game ofchess.The works of C. Berge in 1958 [6] and Ore in 1962 [22] caused domination to become a formal theoretical area of graph theory. But domination did not become an active area until 1977 when a survey paper by Cockayne and Hedetniemi [23] was published. The concept of domination in graphs, with its many variations, is now well studied in graph theory. B. D. Acharya [11] has introduced domination in hypergraph. In semigraph domination has been introduced by S. S. Kamath et. al. [12].Road networks, Traffic routing and density traffic in junction etc. may be studied with the help of domination in semigraph, as well as in disemigraph.

2. Preliminaries

To begin with, some basic concepts and definitions from standard texts [1], [4] and research papers [2], [3] follow.

A *graphG* (*V*,*X*) consists of a finite nonempty set *V* of points together with a prescribed set *X* of unordered pairs of distinct points of *V*.

In a graph*G* (*V*, *X*),$S \subseteq V$ is a *dominating set* of *G* if every vertex of *G* is either in *S*or joined by an edge from some vertex of *S*.

An edge (*u*,*v*) is called an edge from *u* to *v*; we also say that *u* dominates *v*.

A *digraphD* (*V*, *A*) consists of a set *V* of vertices and a set *A* of arcs (directed edges).

For any vertex*v* of a digraph *D* (*V*, *A*), the *reachable setR*(*v*) of *v* is defined as the set of all vertices of *D* reachable from *v*. For a subset *S* of *V*, the reachable set *R*(*S*) of *S* is the set of vertices reachable from some vertex of *S*. A set *S* is a *cover* for *D* if *R*(*S*)=*V*. *Abasis* for *D* is a minimal set *B* of vertices such that *R*(*B*)=*V*; that is, a basis is a minimal cover.

For any vertex *v* of *D*, the *antecedent setQ*(*v*) of *v* is the set of all vertices *u* from which *v*is reachable. For a subset *C* of *V*, the antecedent set *Q*(*C*) of *C* is the set of all vertices of *D* from which some vertex of *C* is reachable. A *contrabasis* for *D* is a minimal subset *C* of *V* such that *Q*(*C*) =*V*.A *duobasis* of *D* is a subset *S* of *V* such that *R*(*S*)=*V*=*Q*(*S*).

A digraph is *strong* if any two of its vertices are mutually reachable.

Let *D* (*V*, *A*) be a digraph. A subset *S* of the vertex set *V* (*D*) is a *dominating set* of *D* if for each vertex *v* not in *S* there exists a vertex *u* in *S* such that (*u*, *v*) is an arc of *D*. Thus a dominating set $S \subseteq V$ is a set of vertices that dominate every vertex in *D*. It may be noted that *V*(*D*) itself is a dominating set of *D*. A dominating set *S* of *D* for which no proper subset of *S* is a dominating set is called a *minimal dominating set* of *D*. The cardinality of a minimal dominating set is called the domination number of *D* and is denoted by $\gamma(D)$.

A *semigraph G* is a pair (*V*, *E*) where *V* is a non-empty set whose elements are called vertices of *G*, and *E* is a set of *n*-tuples, called edges of *G*, of distinct vertices, for various $n \geq 2$ satisfying the following conditions-

S.G.1- Any two edges have at most one vertex in common.

S.G.2- Two edges $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_m)$ are considered to be equal if and only if

i. $m=n$ and

ii. either $u_i=v_i$ for $1 \leq i \leq n$, or $u_i=v_{n-i+1}$ for $1 \leq i \leq n$.

All vertices on an edge of a semigraph are considered to be adjacent to one another.

A *cycle*in semigraph is a closed path. A *cycle space* is a vector space over the field *F*= {0,1} where 1+1=0. A *cycle basis* of a semigraph *G* is a basis for the cycle space of *G* consisting of only cycles.

The *adjacency graph*G_a of a semigraph *G* is a graph with the same vertex set *V*(*G*) and such that two vertices in G_a are adjacent if and only if they are adjacent in *G*. The *distanced (u, v)* between two vertices *u* and *v* in a semigraph *G* is defined as the distance between them in the adjacency graph G_a of *G*. A subset $S \subseteq V$ in *G* is called an *independent set* if noedge is a subset of *S*, and *S* is *e-independent* if no two end vertices of an edge belong to *S*. Further,*S* is *strongly independent* if no two vertices of an edge belong to *S*.

A directed semigraph or *disemigraphD* is a finite set of objects called vertices together with a (possibly empty) set of ordered *n*-tuples of distinct vertices of *D* for various $n \geq 2$, called directed edges or arcs, satisfying the condition-

"For any two distinct vertices *u* and *v*in a disemigraph *D*, there is atmost one arc containing *u* and *v* such that *u* is adjacent to *v* and at most one arc containing *u* and *v* such that *v* is adjacent to *u*".

While drawing a disemigraph in a plane, the initial (or terminal) vertex of an arc which is not a middle vertex of any arc is represented by a thick dot. Middle vertices of arcs are represented by small circles. If a middle vertex is also an initial (or terminal) vertex of an arc, we draw a small tangent to the circle.

If a=$(u_1,u_2,\ldots,u_n)$ is an arc in a disemigraph *D* then a *subarc* of a is an *r*-tupple $(u_{i_1}, u_{i_2}, \ldots u_{i_r})$where $1 \leq i_1 < i_2 < \ldots < i_r \leq n$.The *adjacency digraph* D_a of *D* has *V*(*D*) as vertex set where for any two vertices *u* and *v*, *u* is adjacent to *v* if and only if it is so in *D*. A *s-cycle*(*w-cycle*) in *D* is a closed *s*-path(*w*-path) in *D*. A cycle of odd (even)length is called an *odd* (*even*) *cycle*.

A disemigraph *D* is *simple* if any two arcs in*D* either contain atmost one vertex or all vertices in common.

A disemigraph *D*(*V*,*A*) is called *oriented* if $(u_1,u_2,\ldots,u_n) \in A(D) \Rightarrow$ no u_i, $2 \leq i \leq n$ is adjacent to any u_j, $1 \leq i \leq n-1$ where $j<i$.

A disemigraph $D_1(V_1,A_1)$ is a *subdisemigraph* of a disemigraph *D*(*V*,*A*) if $V_1 \subseteq V$ and $A_1 \subseteq A$. The *induced subdisemigraph* <*U*> of a disemigraph *D* induced by *U* is the subdisemigraph whose vertex set is *U* with arc set $A(<U>)=\{a \cap U: a \in A(D)\}$. A *component* of a disemigraph *D* is a maximal connected subdisemigraph of *D*. The *converse* of a disemigraph *D* is the disemigraphD_c with vertex set *V*(*D*) where $(u_1, u_2, \ldots, u_n) \in A(D)$ if and only if $(u_n, u_{n-1}, \ldots\ldots, u_1) \in A(D_c)$.

A *source*is a vertex which has no arcs directed towards it and can reach all others. A*sink*is the dual concept of source.

Any vertex which has in-degree zero and out-degree greater than zero is said to be a *transmitter*. A *receiver* is the dual concept of transmitter.

A disemigraph *D* is *complete* if for any two distinct vertices *u* and *v* in *D*, at least one of the following holds-

i. *u* is adjacent to *v*.

ii. *v* is adjacent to *u*.

A disemigraph *D* is called *strongly complete* if

i. *D* is complete,

ii. Every vertex in *D* appears as an end vertex of an edge in *D*, and

iii. Any two vertices in *D* are mutually reachable.

A complete oriented semigraph is called a *tournament*. A *strong tournament* is a strongly complete oriented semigraph.

For all other terminology and notations not specifically defined here the reader is referred to [1], [2] and [4].

In this paper only connected, simple and oriented disemigraphs with finite number of vertices will be considered unless mentioned otherwise.

3. Basis in Disemigraph

In the following few terms, particularly the concept of point-basis (which will be simply called a basis) in disemigraph are presented for future reference.

3.1. Definitions

Let D (V,A) be a disemigraph with vertex set V and arc set A. Then for any vertex v of D, $R(v)$ is the *reachable set* defined as the set of all vertices of D reachable from v. For a subset S of V, the reachable set $R(S)$ of S is the set of vertices reachable from some vertex of S.

A subset $S \subseteq V$ is said to be a ***cover*** of D if $R(S)=V$.

A minimal cover of D is said to be a ***point-basis*** (or ***basis***) for D.

The cardinality of a minimum basis set is said to be the ***basis number*** and is denoted by$\delta(D)$. Trivially $1\leq\delta\leq n-m$, where n is the total number of vertices and m is the number of middle vertices in D. For a disconnected disemigraph D a basis set S is the minimal collection of vertices where any vertex of D is either in S or reachable from some vertex of S.

It is observed that a basis set S satisfies the following conditions-

i) All vertices of $D$are reachable from some vertex in S, and

ii) No vertex in S can reach another vertex of S.

Here follows an application of basis in disemigraph.

- Example3.1:

The end vertices of an edge represent the parents and the middle vertices represent their children. The relationship among the members of a number of families can be represented by a disemigraph $D(V,A)$, where $V=\{a,b,c,d,e,f,g,h,i,j,k,l\}$ as given in the diagram bellow. The directions from father to mother is drawn. The disemigraph then has the basis $\{a,f,i\}$ which consists of the guardians of the families, where father or grandfather is assumed as guardian of some family. Here$\delta(D)$=3.

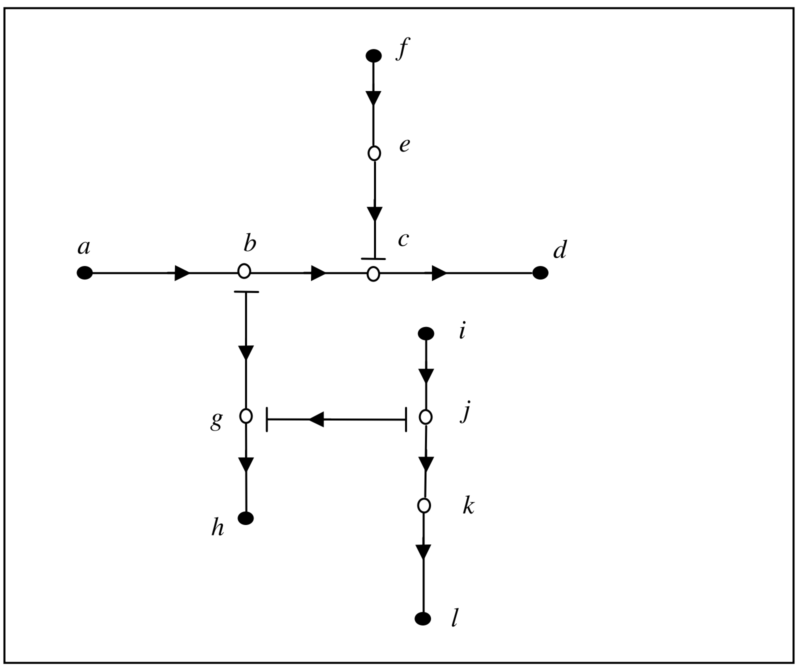

Figure 1

Similarly group discussion among some groups can be represented and studied by a disemigraph with leaders from different groups forming basis for the disemigraph.

It may be noted that all the bases in a disemigraph have the same cardinality.

Analogous to semigraphs the following concepts can be defined.

3.2. Definitions

A set $S \subseteq V$ in D is an *independent set* (Strongly independent set) of D if no two vertices of an arc belong to S. An independent set is said to be the *maximal independent set* if no other vertex can be added to the set without destroying its independence property. An independent set will be the maximum independent set if there exists no other independent set with greater cardinality. The cardinality

of the maximum independent set is said to be the *independence number* of D and is denoted by $\beta(D)$. Here, a simple observation can be made viz., $1\leq\beta\leq n-m$, where n is the total number of vertices and m is the number of middle vertices in D.
A *Strong component* of a disemigraph D is a component of D which is strong in the sense that any two vertices of the component are mutually reachable.
In the following few characterizations of basis set are deduced.

- Proposition 3.1: Every basis in a disemigraph is an independent set.

→ Proof: Let $D(V,A)$ be a disemigraph and S be a basis in D.Then $R(S)=V$ and S is minimal. Suppose $u\in S$ is reachable from $v\in S$. Then $T=S-\{u\}$ also satisfies $R(T)=V$. Thus S is not minimal, which is a contradiction. Hence S is an independent set. Thus every basis set is an independent set.

- Corollary 3.1:$\delta\leq\beta$.
- Corollary 3.2:No two vertices of a basis belong to the same strong component of D.
- Proposition 3.2: Every basis set in adisemigraph consists of at least one vertex from each initial strong component.

→ Proof: Let D be a disemigraph with a basis set S and C be any initial strong component in D. If possible, let$u\in S$but $u\notin C$. Then D does not contain any path from u to Cbecause C is initial. This contradict that S is a basis in D. Hence $u\in C$.

- Corollary 3.3:A basis of a disemigraph D cannot contain more than one vertex from each strong component of D.
- Proposition 3.3: Every disemigraph has a basis.

→ Proof: Let $D(V,A)$ be a disemigraph.
To show that$D$ has a basis.
Let $B=\{S\subseteq V|R(S)=V\}$ ……..(i)
Then $B\neq\phi$ as $R(V)=V$.
If D has an arc i.e. if$D$ is not totally disconnected then V is not minimal with respect to (i). Since D is finite, there exists a minimal $S\subseteq V$ such that $R(S)=V$.
Thus S is a basis of D.

- Corollary 3.4: Every disemigraph contains an independent set.
- Proposition3.4: For a connected disemigraph, all transmitters belong to every basis of the disemigraph.

→ Proof: Let D (V,A) be a connected disemigraph with vertex set V and edge set E. Let S be any basis of D and $T=\{t_1, t_2, …., t_n\}$ be the set of all transmitters of D.
If possible let $t_i\notin S$ for any $t_i\in T$.S being a basis of D, t_i is reachable from some vertex in S, which contradicts the fact that $t_i$is a transmitter. So $t_i\in S$, for any$t_i\in T$.
Thus all transmitters belong to every basis of the disemigraph.

- Corollary 3.5: Every acyclic disemigraph has a unique basis consisting of its transmitters.
- Corollary 3.6: A disemigraph having a source u (say) contains exactly one basis $S=\{u\}$.
- Proposition3.5: If $D(V,A)$ is strongly complete then $|B|=|V|$, where B is the set of all bases in D.

→ Proof: Let D (V,A) be a strongly complete disemigraph with vertex set V and arc set A. Then any two vertices of D are mutually reachable.
Let $v\in V$be any vertex of D. Then any other vertex of D is reachable from v and so $\{v\}$ constitutes a basis in D.
Thus if B is the set of all bases in D then $|B|=|V|$.

- Proposition 3.6: For any tournament, $\delta=\beta=1$.

→ Proof: Trivial.
Analogous to semigraphs, the distance between any two vertices in a disemigraph is defined as follows.

3.3. Definition

Let D (V, A) be a disemigraph. Then for any two vertices u and v in D, *distance* between u and v denoted by $d(u,v)$ is the distance between the vertices in the underlying adjacency digraph D_a.

3.4. Definitions

For any vertex u of D, the *k-reachability set*$R_k(u)$ is $\{v\in V|d(u,v)\leq k\}$, where $d(u,v)$ is the distance from u to v.
A set $S\subseteq V$ is a *k-cover* for D if $R_k(S)=V$.
A set $S\subseteq V$ is a *k-basis* if it is a minimal*k*-cover with no two vertices of S being mutually adjacentand reachable from any other vertex of Swithin k steps.
Like the basis number a *k*-basis number can be defined for a disemigraph which is denoted by δ_k. Obviously, $1\leq\delta_k\leq n-m$, where n is the total number of vertices and m is the number of middle vertices in D.

- Remarks 3.1:

i) All *k*-bases in a disemigraph may not have the same cardinality.
e.g. The following disemigraph $D(V,A)$ with vertex set $V=\{a,b,c,d,e,f\}$ has two 1-bases namely $\{a,c\}$ and $\{f,b,e\}$.

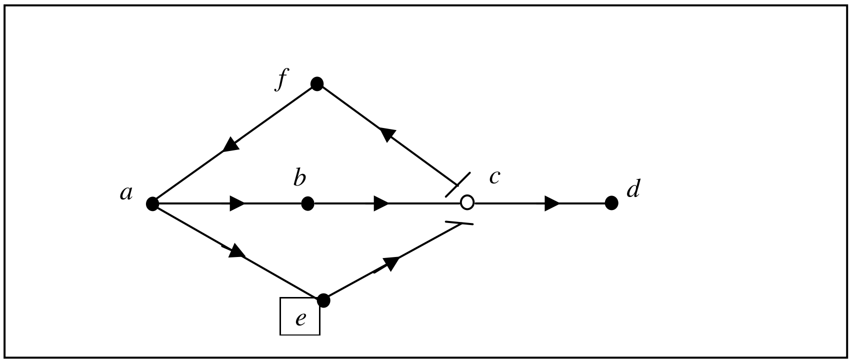

Figure 2

ii)Every tournament in a disemigraph contains a 2-basis.
iii)Every strong tournament in adisemigraph contains a 1-basis.
iv)Every minimal *k*-cover in a disemigraph may not be a *k*-basis.

e.g. In the following disemigraph {1,2,5,6} is a minimal 1-cover but not a 1-basis.

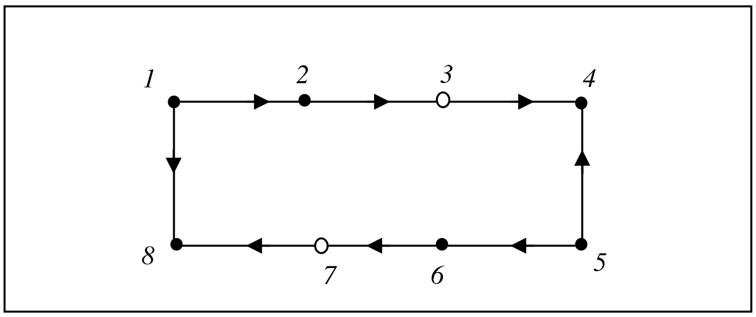

Figure 3

v) Every disemigraph may not have a *k*-basis. See Figure 3
It will not be inappropriate to cite the following results that hold in digraph setting but may not hold in a disemigraph setting.
(i)[1] Every digraph with no odd cycles has a 1-basis.
This result is not necessarily true in disemigraph as can be observed from the figure 4, which shows a disemigraph without having any odd cycle as well as any 1-basis.

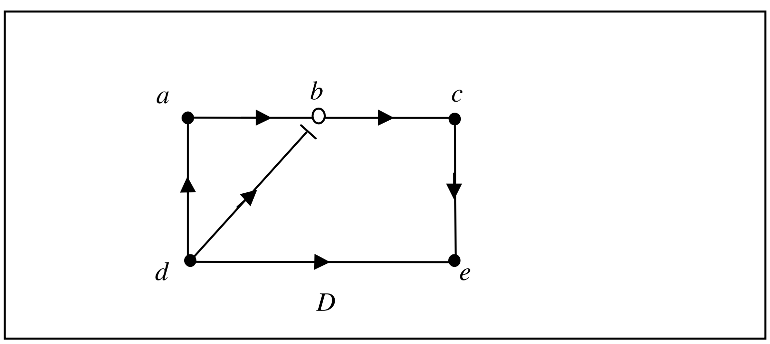

Figure 4

(ii) [1] Every acyclic digraph has a 1-basis.
The disemigraph*D* displayed in figure 4 is acyclic but does not contain any 1-basis.

- Proposition 3.7: For any disemigraph *D*, $\delta(D) \leq \delta_k(D)$.

→ Proof: From the definition of basis and k-basis, it is obvious that the basis set is a subset of some *k*-basis set. Thus the result follows.

3.5. Definition
A vertex *w* in *D*(*V*, *A*) is an *r-king* if $d(w,v) \leq r$, $\forall\, v \in V$.
e.g. In Figure 4,*d* is a 2-king.
From the above definition remarks can be made as follows-

❖ Remarks 3.2:

i) Every disemigraph may not have an$r$-king. Figure 1 may be referred.
ii) Every tournament T contains a 2-king. And, any vertex with highest out degree (i.e. score) is a 2-king.
iii) Every r-king is an$r$-basis.

3.6. Definitions

For any vertex v of D, the *antecedent set* $Q(v)$ of v is the set of all vertices from which v is reachable. For any set $S \subseteq V$, the antecedent set $Q(S)$ is the set of all vertices of D from which some vertex of S is reachable.
A minimal subset C of V is called a *contrabasis* if $Q(C)=V$.
A *duobasis* of D is a subset $S \subseteq V$ such that $R(S)=V=Q(S)$.
The properties of contrabasis are analogous to those of basis where each arrow is replaced by the opposite counterpart. Thus the contrabasis set of a disemigraph is same as the basis set of the disemigraph obtained by reversing the arrows of the original disemigraph.
Again few remarks are in order-

❖ Remarks 3.3:

i)A duobasis has always cardinality equal to one.
ii) Contrabasis and duobasis are both independent sets.
iii) Every disemigraph possesses both a basis and contrabasis but it may not have a duobasis.
e.g. In Figure 3, {1,5} is a basis and {4,8} is a contrabasis for the disemigraph but it does not have any duobasis.
iv) A source is a basis of cardinality one and a sink is a contrabasis of cardinality one.
v) Every strong tournament contains a duobasis.
vi) A disemigraph having source or sink as well as transmitter or receiver cannot possess a duobasis.
e.g.Figure 3 and Figure 4 are referred.

➢ Proposition 3.8: A disemigraph contains a duobasis if and only if it is strong.

→ Proof: Let D be a disemigraph with vertex set V.

We assume that D contains a duobasis, say S, such that $R(S)=V=Q(S)$.
Then any vertex in V is both reachable from and reachable to ($\because R(S)=Q(S)=V$) to the vertex of S ($\because |S|=1$). Thus any two vertices of D are mutually reachable. Hence D is strong.
Conversely, Let D be strong.
Then D has a spanning cycle and so any two vertices in D are mutually reachable. Thus for any vertex $v\in V$,v is reachable to and reachable from any other vertex of D. Hence $S=\{v\}$ forms a duobasis.

- Corollary3.7: If B is the set of all duobasis in a strong (strongly complete) disemigraph D, then $|B|=|V|$.
- Corollary 3.8: A disemigraph contains a duobasis if and only if it has a spanning cycle.

➢ Proposition 3.9: All the receivers of a disemigraph D belongs to every contrabasis of D.

→ Proof: Let $D(V,A)$ be a disemigraph with a contrabasis C and let $R=\{r_1,r_2\ldots,r_s\}$ be the set of all receivers of D. If possible, let $r\notin C$ for any $r\in R$.C being the contrabasis in D, r is reachable to some vertex of C in D which guarantees that od(r)≥1. This is a contradiction to the fact that r is a receiver in D. Thus $r\in C$, for any receiver r of D. Hence the result follows.

- Corollary 3.9: If a disemigraph D has a sink t, then D contains the contrabasis $C=\{t\}$ only.

4. Domination in Disemigraph

In this section an attempt has been made to develop the concept of domination (i.e. vertex domination) in disemigraphs.

4.1. Definition

Let $D(V,A)$ be a disemigraph.
A subset S of V is said to be a *vertex dominating set* of D if for each vertex $v\in V\backslash S$ there exists a vertex $u\in S$ such that (u,v) is an arc or subarc in D. A dominating set will be simply referred to as a dominating set.
A dominating set S is a *minimal dominating set* if no proper subset of S is a dominating set. A dominating set S is a *minimum dominating set* if there is no dominating set with smaller number of elements(vertices).The cardinality of a minimum dominating set is said to be the *domination number* of D and is denoted by $\gamma(D)$.
Note that $V(D)$ itself is a dominating set of D. If S is a dominating set in D then every vertex in V is reachable from some vertex of S within one step i.e. S dominates every other vertex in D.
Recall that in an arc or sub arc $(u_1, u_2,\ldots,u_n)$, u_i is adjacent to u_j for $1\le i\le j\le n$.Here we say that u_i dominates u_j, for $1\le i\le j\le n$.

❖ Example 4.1:

In the following disemigraph$D(V,A)$,$S=\{a,b,e\}$ is a minimum dominating set and so$\gamma(D)=3$.

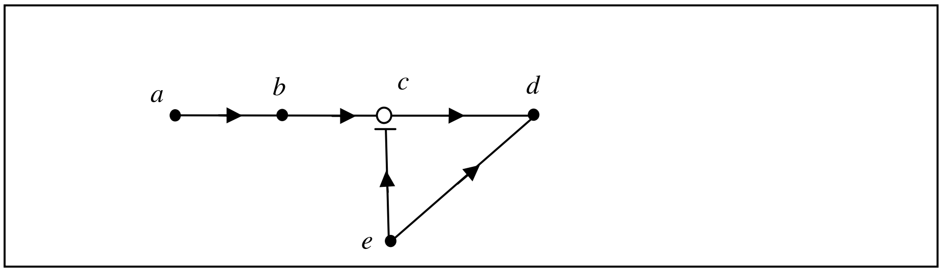

Figure 5

From the above definition of dominating set few remarks follow.

- Remarks 4.1:

i) Every disemigraph contains a dominating set.
ii) Every basis set is a subset of some dominating set.
iii) Every 1-basis is a dominating set and so $\delta_1=\gamma$.
iv)$\delta\leq\beta\leq\gamma$.
v)A dominating set may or may not be an independent set.
vi) A disemigraph may have many minimal dominating set of different sizes but every disemigraph can have only one minimum dominating set.
vii) Every tournament or a strong tournament contains a dominating set of maximum cardinality 2.

4.2. Definitions

Let $D(V,A)$ be a disemigraph. A dominating set S is said to be *total dominating set*if the induced subdisemigraph $\langle S\rangle$ has no isolated vertices.
The cardinality of a minimum total dominating set is called the *total domination number* of D and is denoted by $\gamma_t(D)$.
e.g. For the disemigraph D given in Figure 5,$S_t=\{a,b,c,e\}$ is a minimum total dominating set and so $\gamma_t(D)=4$.
In the following two types of connected dominating set are defined depending on whether the dominating set is strong or weak.

4.3. Definitions

A dominating set S of a disemigraph$D$ is said to be a *weakly connected dominating set*(*wc*-dominating set) if the induced subdisemigraph $\langle S\rangle$ is weakly connected.The cardinality of a minimum *wc*-dominating set is the *weakly connected domination number* of D and is denoted by $\gamma_{wc}(D)$.
A dominating set S of a disemigraph D is said to be a *strongly connected dominating set* (*sc*-dominating set) if the induced subdisemigraph $\langle S\rangle$ is strongly connected. The cardinality of a minimum *sc*-dominating set is the *strongly connected domination number* of D and is denoted by $\gamma_{sc}(D)$.

- Example 4.2:

Let $D(V,A)$ be a disemigraph with vertex set $V=\{a,b,c,d,e,f,g,h,i,j\}$. Here$D$ has a*wc*-dominating set $S_{wc}=\{a,c,e,f,g\}$ and sc-dominating set $S_{sc}=\{a,c,e,f,g,h\}$ which areminimum, thus $\gamma_{wc}(D)=5$ and $\gamma_{sc}(D)=6$. Here D has a dominating set $S=\{a,c,e,g\}$ and so $\gamma=4$

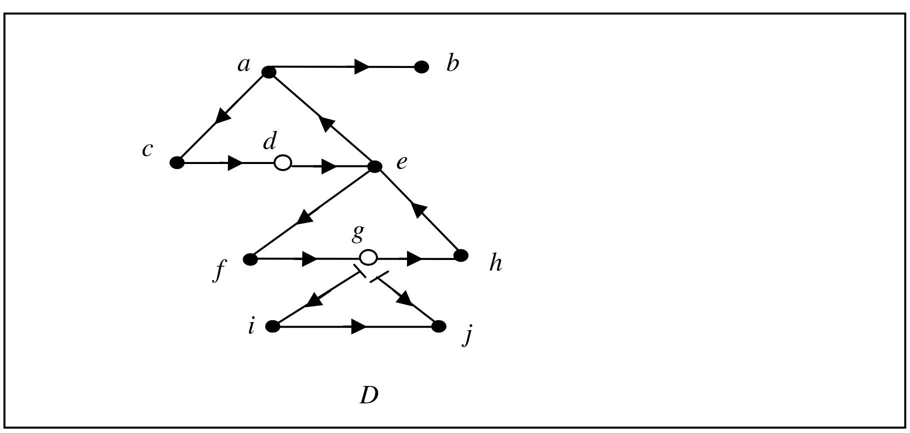

Figure 6

From the above definitions of total and connected dominating set followings remarks are immediate.

❖ Remarks 4.2:

i) Every disemigraph contains a total dominating set and a connected dominating set.

ii) Every connected dominating set is a total dominating set but the converse is not necessarily true.

iii) $\gamma \leq \gamma_{wc} \leq \gamma_{sc}$

4.4. Definition

A dominating set S is called *open dominating set* if for all $v \in V(D)$, there is $u \in S$ such that (u,v) is an arc or subarc in D.
The cardinality of a minimum open dominating set is called the *open domination number* of D and is denoted by $\gamma_o(D)$.
e.g. For the disemigraph D given in Figure 6, $S_o=\{a,c,e,f,g\}$ is a minimum dominating set and so $\gamma_o(D)=5$.
Now, the following remarks about an open dominating set are quite natural.

❖ Remarks 4.3:

i)Every disemigraph may not have an open dominating set.
e.g. Let $D(V,A)$ be a disemigraph with vertex set $V=\{a,b,c\}$ and arc set $A=\{(a,b), (a,c)\}$. Then D has a dominating set $S=\{a\}$ but it does not have any open dominating set.

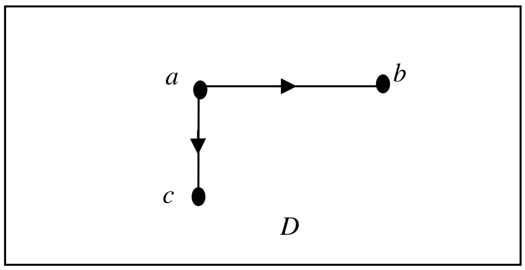

Figure 7

ii) A disemigraph containing an open dominating set cannot have a source (resp. transmitter) and sink (resp. receiver).

iii) Every disemigraph containing open dominating set contains a cycle.

iv) A tournament may not have an open dominating set but a strong tournament always has an open dominating set.

Now few results on relationships among independent set, basis set and dominating set are established in the following.

➢ Proposition 4.1: Every basis setis a subset of some dominating set.

→ Proof: Let $D(V,A)$ be a disemigraph.

Let B be a basis set and S be any dominating set.
For any $v\in V$, there exists some $u\in B$ such that v is reachable from u. And obviously some $w\in V$ exists such that $(u,w)\in A$ which implies that u dominates w in D. So u is in some dominating set, say, S i.e. $u\in S$.
Thus every basis set is a subset of some dominating set i.e. $B\subseteq S$.

- Corollary 4.1: A basis set is a subset of all kinds of dominating set provided any such dominating set exists.

➢ Proposition 4.2: Every maximal independent set is a dominating set.

→ Proof: Let $D(V,A)$ be a disemigraph and $S\subseteq V$ be a maximal independent set.

If S does not dominate D, then there is at least one vertex which is neither belongs to S nor is adjacent to any other vertex in the set S. Such a vertex can be added to the independent set without destroying its independence property. But then the maximality condition of the independent set S will be violated. Hence the result follows.

- Corollary 4.2: An independent dominating set is the same as a maximal independent set.
- Corollary 4.3: $\gamma(D)\leq\beta(D)$ for any disemigraph D.

In the following some characterizations of the various dominating sets are derived.

➢ Proposition 4.3: For any disemigraph $D(V,A)$, $1\leq\gamma_d(D)\leq(n-m)$, where γ_d is the domination number of any kind, $n=|V|$ and m is the number of middle vertices in D.

→ Proof: Obviously $\gamma_d(D)\geq 1$. Consider an arc $(u_1,u_2,\ldots,u_t)$, then all the middle vertices u_i for $1<i<t$ are dominated by u_1. So clearly $\gamma_d(D)\leq(n-m)$. Hence $1\leq\gamma_d(D)\leq(n-m)$.

- Corollary 4.4: If D is a tournament then $\gamma_d=2$.

➢ Proposition 4.4: For any disemigraph $D(V,A)$
$\gamma_t(D)\leq n-\Delta^+(D)+1$, where $\Delta^+(D)=\max\{od(v): v\in V\}$

→ Proof: Let $D(V,A)$ be a disemigraph of order n.

Let v be any vertex in D with maximum out-degree p (say).
So $\Delta^+(D)=p$.
Now v will dominate exactly p vertices in D and for the remaining $n-p-1$ numbers of vertices excepting v a maximum of $n-p-1$ vertices can be found to form a dominating set S (say) for D along with v.

If $id(v)\geq 1$ then S will be a total dominating set. Also if $id(v)=0$ then S must contain at least one vertex from$p$ vertices to satisfy the criteria for a dominating set to be a total dominating set (otherwise S will contain v as an isolated vertex) along with v and the remaining$n-p-1$ vertices.

Thus $|S|\leq 1+n-p-1+1$

$=n-p+1$

$\Rightarrow\gamma_t(D)\leq n-\Delta^+(D)+1$.

- Corollary 4.5:$\gamma(D)\leq n-\Delta^+(D)$.

Here we reproduce the following result due to S. Arumugam et al. which will be required to prove the next proposition.

Theorem 2.5: [15] If $D=(V,A)$ is weakly connected digraph of order n with $\delta^-(D)>0$, then $\gamma_o(D)=n$ if and only if D is a dicycle, where $\delta^-(D)$ is the minimum in-degree of D.

- Proposition 4.5: If $D(V,A)$ is a disemigraph of order n with m number of middle vertices and with no transmitter or receiver then $\gamma_o(D)=n-m$ if and only if$D$ is a directed cycle.
- → Proof: Let $D(V,A)$ be a disemigraph of order n with m number of middle vertices .

By proposition 4.3, $1\leq\gamma_o(D)\leq(n-m)$.

Case i) $m=0$

Then D becomes simply a digraph which is connected and with $\delta^-(D)>0$ ($\because D$ is without any transmitter),so by Theorem 2.5:[15] the result is immediate.

Case ii) $m\geq 1$

Let D be without any transmitter or receiver and $\gamma_o(D)=n-m$. It is required to show that$D$ is a directed cycle. If possible, let$D$ is not a directed cycle. Then D contains at least one transmitter or receiver, which is a contradiction. Hence D is a directed cycle.

Conversely, let$D$ be a directed cycle. Then it is obvious that $\gamma_o(D)=n-m$.

Nowan application of domination is discussed in the following-

- Application 4.1:

Let us consider a region where a State Transport Corporation conducts running of its buses to various destinations via some stoppages to cover the whole region by some route maps. This situation can be represented by disemigraph where each stoppage represents a vertex; each edge represents the road way with the stoppages other than the initial and terminal ones as middlevertices. The direction in an edge is from the starting point of journey to the terminal one. This is the example of various domination types by which the Corporation can accommodate and provide best service to the public.

Following conjectures and problems are posed for further study and solutions.

- Conjecture4.1: For any disemigraph $D(V,A)$, $\gamma_{wc}(D)\leq 2\beta$.
- Conjecture4.2: $1\leq\gamma\leq\gamma_t\leq\gamma_{wc}\leq\gamma_o\leq n$, if the specific dominating set exists.
- Problem 4.1:To characterize all the disemigraphs for which $\delta=\beta=\gamma$.
- Problem 4.2: To investigate the bounds for strongly connected domination number γ_{sc}.

5. References

i. Harary, F. (2001). Graph Theory, Narosa Publishing House.
ii. Sampathkumar, E. (May 2000). Semigraphs and Their Applications, Report on the DST Project.
iii. Das, P. & Hoque, Hamida Aktara (March-2012).A Note on Tournaments in Disemigraph, Int. J. Math. Sc. & Engg. Appls. Vol.6, No.II, pp. 299-307.
iv. Parthasarathy, K. R. (1994). Basic Graph Theory, Tata McGraw-Hill Publishing Company Limited, New Delhi.
v. Sampathkumar, E. (June 2003) Semigraphs, SERC research Highlights, Department of Science and Technology, New Delhi-16, 370-389.
vi. Berge, C. (1962). The Theory of Graphs and its Applications, Methuen and Co., London.
vii. Berge, C. (1973).Graphs and Hypergraphs, North Holland, Amsterdam.
viii. König, Dénes (1936). Theorie der endlichen und unendlichen Graphen. Akademische verlagsges., Leipzig. Repr. Chelsea, New York, 1950. English trans. by Richard McCoart, Theory of Finite and Infinite Graphs. Birkhäuser, Boston, 1990.
ix. Diestel, Reinhard (2005). Graph Theory, Electronic Edition, New York.
x. Lee, Changwoo (1998). Domination in Digraph, J. Korean Math. Soc. 35, No. 4, pp. 843-853.
xi. Acharya, B.D. (2007) Domination in Hypergraphs, AKCE J. Graphs Combin., 4, No.2, pp 117-126.
xii. Kamath, S. S. & Baht, R. S. (2003). Domination in semigraphs, Electronic Notes in Discrete Mathematics, pp.106-111.
xiii. Kamath, S. S. & Hebbar, Saroja R. (2003). Strong and Weak domination full sets and domination balance in Semigraphs, *Electronic notes in discretemathematics* 15, pp. 112.
xiv. Lee, C. On the domination number of a digraph, Ph.D. Dissertation, Michigan University.
xv. Aurumugam, S., Jacob, K., & Volkmann, L. (2007). Total and Connected domination in digraphs, Australasian Journal of Combinatorics, Vol.39, pp.283-292.
xvi. Foulds, L. R. (1993). Graph Theory Applications, Narosa Publishing House.
xvii. Sampathkumar, E. (1996). B. Systems, Proc. National Workshop on Graph Theory and its Applications, Manonmanian Sundarnar University, Tirunelveli, 173-185, Tata McGraw Hill, New Delhi,.

xviii. Kamath, S. S. & Hebbar, S.R. (2010). Domination in Critical Semigraphs, Domination in Semigraphs (part III), Lecture Notes on National Workshop on Semigraphs, College of Engineering, Pune.

xix. Thakkar, D.K.&Prajapati, A. A. (2013). Consecutive Adjacent Domination Number in Semigraphs, Journal of Computer and Mathematical Science, vol.4(I),69-74.

xx. Sampathkumar, E. &Walikar, H.B.(1979). The connected domination number of a graph, J. Math. Phy. Sci. 13,607-613.

xxi. Harary, F., Norman, R.Z. & Cartwright, D. (1965). Structural Models: An Introduction to the Theory of Directed Graphs. Wiley, New York.

xxii. Ore, O. (1963). Graphs and their Uses, Random House, New York.

xxiii. Cockyne, E.J. & Hedetniemi, S.T. (1977). Towards a theory of domination in graphs Networks, pp. 247-261.